



Improve refinement approach iterative method for solution linear equation of sparse matrices

Desi Vinsensia¹, Yulia Utami², Fathia Siregar³, Muhammad Arifin⁴

¹Management Informatics, STMIK Pelita Nusantara, Medan, Indonesia

²Informatics Technique, STMIK Pelita Nusantara, Medan, Indonesia

^{3,4} Information Technology, STMIK Pelita Nusantara, Medan, Indonesia

Article Info

Article history

Received : Dec 15, 2023

Revised : Jan 14, 2023

Accepted : Jan 28, 2023

Keywords:

Rate converge;

Refinement;

Sparse matrices;

Spectral radius

Abstract

In this paper, systems of linear equations on sparse matrices investigated through modified improve method using Gauss-Seidel and successive overrelaxation (SOR) approach. Taking into adapted convergence rate on the Improve refinement Gauss-seidel outperformed the prior two Gauss-Seidel methods in terms of rate of convergence and number of iterations required to solve the problem by applying a modified version of the Gauss-Seidel approach. to observe the effectiveness of this method, the numerical example is given. The main findings in this study, that Gauss seidel improvement refinement gives optimum spectral radius and convergence rate. Similarly, the SOR improved refinement method gives. Considering their performance, using parameters such as time to converge, number of iterations required to converge and spectral radius level of accuracy. However, SOR works with relaxation values so that it greatly affects the convergence rate and spectral radius results if given greater than 1.

Corresponding Author:

Desi Vinsensia,

Management Informatics

STMIK Pelita Nusantara

Jl. Iskandar Muda, No.1, Medan, Sumatera Utara, Indonesia, 20154

Email: desivinsensia87@gmail.com

This is an open access article under the [CC BY-NC](https://creativecommons.org/licenses/by-nc/4.0/) license.



1. Introduction

The area of numerical analysis provides increased attention to mathematical derivatives, both in terms of descriptive and analytical methods to obtain numerical solutions to mathematical problems. The interest of numerical analysis method is the constructive method that showed how to develop the solution of mathematical problems in this case discussing the system of linear equations. Given a matrix $A \in \mathbb{R}_{n \times n}$ and $x, b \in \mathbb{R}$ is a linear system with symbolized as $Ax = b$. A collection of linear systems is known as a system of linear equations. In order to obtain the solution of linear equations, the direct method and the iterative method are two methods. The direct method is a method with less round-off errors, so it gives the exact solution in a limited number of basic row operations. In general, the direct method works with a number of steps and then through operations the exact solution is given. The numerical solution of a system of linear equations concluded that the direct method is inappropriate for solving a large number of equations in a system [1], particularly when the coefficient matrices are sparse, i.e. when most of the elements in the matrix are zero, in contrast to the LU decomposition method.

The exact solution of this method depends on rounding that is inefficient when compared to the iteration method which uses the tolerance value as an operating input parameter thus the solution of steps to improve the accuracy of the solution obtained after the computing process. In certain cases, especially in large systems, the iteration method is more suitable for used [2]. However, the iteration method requires smaller storage space if it has sparse properties in the matrix coefficients [3]. There are two types of matrices based on the linear system, namely dense and sparse. Dense matrices have few zeros, and the order of the matrix tends to be relatively small. Sparse matrices consist of a few non-zero elements. The sparse linear system is derived from the concept of the finite difference method for solving a system of second-order ordinary differential equations. Several researches widely used sparse matrices, in various application such as speech recognition, computer vision [4], [5], artificial neural networks [6],[7], deep learning workloads[8], biology [9], chemistry [10], etc. The solution of linear sparse equation system, direct method and iteration method can be used, such as Jacobi iteration method, Gauss Seidel and also Successive Over Relaxation (SOR) method [11], [12].

Direct methods have a finite set of procedures that give an exact solution. They are robust but difficult to parallelise, consume memory and also more cost. Instead, the iterative methods have a sequence of approximation solutions, starting with an initial guess and improving the solution until it converges to close an exact solution. Several studies developed sparse matrices in evaluating the storage performance of Compressed Sparse Row (CSR)[13] and Block Compressed Sparse Row (BCSR) in Message Passing Interface (MPI) [14]. While [15] uses a sparse matrix divided into two segmentations based on the regularity of memory access patterns, where each segmentation is stored in a format that matches its memory access pattern. to build a predictive model to automatically determine the partition threshold on a per basis matrix. In this paper extended version of our earlier work [16] which iteration methods focusing on Gauss-Seidel method, and SOR as methods used in solving the solution of sparse matrices analyzes the convergence criteria for both iteration methods. This study is expected to be one of the references in determining the solution of sparse linear systems with large matrices. The outline of this paper is as follows: Section 2 contains a comprehensive coverage of the methods used in this research, Gauss-Seidel and SOR. Section 3 discusses the analysis of the convergence results of Gauss-Seidel and SOR and discussion with numerical example. Section 4 is the conclusion.

2. Methods

In this section using iterative method with Gauss Seidel and SOR method. The derivation of the formulation is described and the convergence properties of the two formulas are analyzed. In this study, refinement gauss seidel (RGS) is introduced, which is an expansion of gauss seidel reduction. Improved refinement gauss-seidel (IRGS) is a reduction expansion of the refinement gauss-seidel (RGS) formulation.

2.1 Gauss-Seidel (GS)

The utilization of Gauss-Seidel methods conducted by [17] describes the implementation and performance of an efficient parallel Gauss-Seidel algorithm that developed for irregular and sparse matrices from power system applications. This algorithm is inherently sequential. However, given the special ordering of sparse matrices, it is possible to eliminate many of the data dependencies caused by priorities in the calculations. A method of two-part matrix ordering was developed - first partitioning the matrix into diagonally bounded blocks using a dioptic technique and then performing data coloring on the last diagonal block using a graph coloring technique. In addition, [18] used iterative solving for sparse linear equations by symmetric Gauss-Seidel method to propose Computational Fluid Dynamics (CFD). The method is proposed to solve the right handed equations in incompressible fluid flow solver. Following [19], consider sparse linear system form $Au = f$, where f is a given n -dimensional real vector, u is a vector to be determined, and $A = (a_{ij})$, is a nonsingular real matrices of n -order. Given $-E$ as upper and $-F$ lower triangular of A [20], then $M = D - E$ and $N = F$

$$Au = f \quad (1)$$

$$\text{Becomes } Du + Au = Du + f \quad (2)$$

$$\text{Consider by } u^{(k+1)} = (D - E)^{-1}Fu^k + (D - E)^{-1}f \quad (3)$$

Equation (3) resolved forward substitution as follows:

$$u^{[k+1]} = M^{-1}Nu^{[k]} + M^{-1}f \quad (4)$$

Consider of splitting of A as follows $A = D - E - F$; with substitution split of A , obtained $Du + (D - E - F)u = Du + f$. By modifying (1) as follows:

$$u = u + (D - E)^{-1}(f - Au) \quad (5)$$

In an iterative, u on the left and right sides of equation (4) can be derived:

$$u^{(k+1)} = \tilde{u}^{(k+1)} + (D - E)^{-1}(f - A\tilde{u}^{(k+1)}) \text{ extend this formulation as follows:}$$

$$u^{(k+1)} = (D - E)^{-1}F\tilde{u}^{(k+1)} + (D - E)^{-1}f \quad (6)$$

By rearranging and simplifying eq. (6) yields

$$u^{(k+1)} = ((D - E)^{-1}F)^2u^k + (I + (D - E)^{-1}F)(D - E)^{-1}f, \text{ substitution (6) Also we get}$$

$$u^{(k+1)} = ((D - E)^{-1}F)^3u^k + [(I + (D - E)^{-1}F) + ((D - E)^{-1}F)^2](D - E)^{-1}f \quad (7)$$

Equation (7) improve refinement Gauss Seidel. Then, by rearrange and simply expanding eq. (7), we get:

$$u^{(k+1)} = ((D - E)^{-1}F)^4u^k + [(I + (D - E)^{-1}F) + ((D - E)^{-1}F)^2 + ((D - E)^{-1}F)^4](D - E)^{-1}f \quad (8)$$

Equation (8) is second improve refinement Gauss Seidel. If we do similar to m^{th} , for general we get the form of m -refinement of Gauss-Seidel derivation.

Convergence Gauss-Seidel (GS)

Theorem 1. Let A is strictly diagonally dominant (SDD) matrices, Gauss-Seidel converges for all u^0 .

Proof: The iteration converges if satisfy: $\rho(I - (D - E)^{-1}A) < 1$ (9)

Let λ is a eigen value, $I - (D - E)^{-1}A$ then,

$$(D - L)u - Au = \lambda(D - E)u \quad (10)$$

$$-\sum_{j=i+1}^n a_{ij}u_k = \lambda \sum_{j=1}^i a_{ij}u, 1 \leq i \leq n \quad (11)$$

Then rearranged and simplify (11) and let for the absolute symbol given in the both side, as follows:

$$|\lambda a_{ii}u_k| \leq \sum_{j=i+1}^n |a_{ij}| |u_k| + |\lambda| \sum_{j=1}^i |a_{ij}| |u_k| \quad (12)$$

Since u is eigenvector, $u \neq 0$. Consider $\|u\|_\infty = 1$. Used k such that $|u_k| = 1$ and $|u_j| \leq 1$ for all $j \neq k$.

Hence:

$$|\lambda| \leq \frac{\sum_{j=i+1}^n |a_{ij}|}{|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}|} = \frac{\sum_{j=i+1}^n |a_{ij}|}{|a_{ii}| - \sum_{j=i+1}^n |a_{ij}| - \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^n |a_{ij}|} = \frac{\sum_{j=i+1}^n |a_{ij}|}{(|a_{ii}| - \sum_{j=i+1}^{i-1} |a_{ij}| - \sum_{j=1}^{i-1} |a_{ij}|) + \sum_{j=i+1}^n |a_{ij}|} < 1 \quad (13)$$

Therefore, Gauss-Seidel converges for every initial predictor $u^{(0)}$.

Theorem 2. Let A is a strictly dominant diagonal (SDD) matrices, then improve refinement Gauss Seidel converges for initial predictor $u^{(0)}$.

Proof. Let U is the exact solution. Since A is *matrice SDD*, Gauss-Seidel, and define Gauss Seidel converges. Let $\tilde{u}^{(u+1)}$ converges to U . Such that, using norm in both of methods, as follows:

$$\begin{aligned} \|u^{(k+1)} - U\| &\leq \|\tilde{u}^{(k+1)} - U\| + \|(D - E)^{-1}\|(f - A\tilde{u}^{(n+1)})\| \rightarrow \|U - U\| + \|(D - E)^{-1}\|\|f - AU\| \\ &= 0 + \|(D - E)^{-1}\|\|b - b\| = 0 + 0 = 0 \end{aligned}$$

Therefore $u^{(k+1)}$ converges to U and $\rho(((D - E)^{-1}F)^3) = (\rho((D - E)^{-1}F))^3 < 1$.

Such that Refine Gauss-Seidel converges.

Theorem 3. Define A as simmetry definite positive (SPD) matrices. Then Gauss Seidel convergen for any initial approximation $u^{(0)}$.

Proof. Consider $A = D - E - E^T$, with $F = E^T$ and $A = A^T$ then A then symmetrical matrice. Such that $B_{GS} = (D - E)^{-1}E^T$. We will define that $\rho(B_{GS}) < 1$, with ρ = spectral radius.

Let be λ is a eigen value of B_{GS} and u is a vector eigen. Then $(D - E)^{-1}E^T u = \lambda u \Rightarrow E^T u = \lambda(D - E)u$. Hence $u^H E^T u = \lambda u^H (D - E)u$, with $u^H = (u^*)^T$.

It simplifies to:

$$u^*Au = (1 - \lambda)u^*(D - E)u \quad (14)$$

With transpose conjugate on both side of (14):

$$u^*Au = (1 - \lambda)u^*(D^T - E^T)u \quad (15)$$

From (14) and (15) obtained:

$$\rho((D - E)^{-1}F^T) = \rho((D - E)^{-1}F) < 1 \quad (16)$$

Therefore, the Gauss Seidel method converges if matrices A is a symmetry positive definite matrice (SPD).

Theorem 4. If A is matriks $-M$, then iterative method Improve refinement Gauss Seidel (IRGS) converges for every initial approximation $(x)^0$.

Proof. If A is a M-matrice, then spectral radius Gauss Seidel less than 1. So $\rho((D - E)^{-1}F) < 1$. $\rho(((D - E)^{-1}F)^3) < 1$.

Such that improve refinement Gauss-Seidel converges.

Similarly way, for general refinement Gauss-Seidel converges $(k + 1)$ times fast as Gauss-Seidel method [21].

2.2 Successive Overrelaxation (SOR)

The Successive Overrelaxation (SOR) method accelerates convergence better than Gauss-Seidel. SOR is also useful for solving partial differential equation-based linear systems. SOR have the relaxation value known as φ . Splitting matrices A with the following equations:

$$A = \varphi D - \varphi E - \varphi F + D - D \quad (17)$$

By substituting equation (9) into (3), we obtain:

$$u_i^{(k+1)} = (D - \varphi E)^{-1}[(1 - \varphi)D + \varphi F]u^{(n)} + \varphi(D - \varphi E)^{-1}f \quad (18)$$

Similar to eq. (8), by following the similar method of rearranging and simplifying eq. (17) and eq.(18)

$$\begin{aligned} \text{We Obtained: } u^{(k+1)} &= [(D - \varphi E)^{-1}[(1 - \varphi)D + \varphi E]]^3 u^{(k)} + [I + (D - \varphi L)^{-1}((1 - \varphi)D + \varphi F) + \\ &+ [(D - \varphi E)^{-1}(1 - \varphi)D + \varphi F]^2] \varphi(D - \varphi E)^{-1} \varphi b \end{aligned} \quad (19)$$

Equation (19) improve refinement-SOR. Then, by rearrange and simply expanding equation before we get:

$$u^{(k+1)} = [(D - \varphi E)^{-1}[(1 - \varphi)D + \varphi E]]^4 u^{(k)} + [I + (D - \varphi L)^{-1}((1 - \varphi)D + \varphi F) + [(D - \varphi E)^{-1}(1 - \varphi)D + \varphi F]^2] \varphi(D - \varphi E)^{-1} \varphi b \quad (20)$$

If we do similar to m^{th} , for general we get the form of m –refinement of SOR derivation.

Convergence Successive Overrelaxation (SOR)

Theorem 5. [22] If $A \in \mathbb{C}^{n,n}$ is strictly diagonally dominant), then SOR method converges with boundary $0 < \varphi \leq 1$.

Proof: The Gauss-Seidel method, which is defined as matrices that are strictly diagonally dominant, implies following:

$$u_i^{(k+1)} = \frac{1}{a_{ii}} [b_i - \sum_{j=1}^{i-1} a_{ij} u_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} u_j^{(k)}] \text{ then we can formulate SOR with substitute}$$

the relaxatiton value φ as follows:

$$u_i^{(k+1)} = (1 - \varphi)u_i^{(k)} + \frac{\varphi}{a_{ii}} [b_i - \sum_{j=1}^{i-1} a_{ij} u_j^{(k+1)} - \sum_{j=i+1}^n a_{ij} u_j^{(k)}] \text{ else, it can be written as follows:}$$

$$u_i^{(k+1)} = (1 - \varphi)u_i^{(k)} + \varphi x_{GS}^{(k+1)} = u_i^{(k)} + \varphi(u_{GS}^{(k+1)} - u_i^{(k)}). \text{ The relationship between } u_i^{(k)} \text{ and}$$

$u_{GS}^{(k)}$ can be demonstrated. The SOR method's solution is a convex set for all values of $0 < \varphi \leq 1$.

Let $u^{(k+1)}$ be the $(k + 1)^{th}$ approximation to the solution of (1) from SOR methods. Suppose u_E is the exact solution of linear system (1), then $\|u^{(k+1)} - u_E\| = \|u^{(k+1)} + \varphi(I - \varphi U)^{-1}(f - Au^{(k+1)}) - u_E\| \leq$

$\|u^{(k+1)} - u_E\| \|f - Au^{(k+1)}\| \|\varphi(I - \varphi U)^{-1}\| \rightarrow 0$ then $\|u^{(k+1)} - u_E\| \rightarrow 0$. Hence the refinement SOR is converges. Similar way, it's clearly shown that for the general refine converges $(k + 1)$ times fast as SOR method.

2.3 Numerical Example

Consider linear equation with has coefficient a strictly dominant diagonal (SDD) matrices and a positive definite symmetry (SPD) matrices as follows:

$$6x_1 + 2x_1 + 2x_3 = 5$$

$$2x_1 + 8x_2 + 2x_3 = 6$$

$$2x_1 + 2x_2 + 10x_3 = 7$$

For tolerance 10^{-5} , we will show the number of iteration through convergence rate, spectral radius, and number of iterations of the Gauss Seidel and refine SOR methods.

Solution:

Tabel 1.
Numerical Result for Gauss-Seidel

n	Gauss -Seidel			Refinement Gauss-Seidel			Improve Refinement Gauss Seidel		
	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
1	0.8333	0.5417	0.4250	0.5111	0.5160	0.4946	0.4965	0.5022	0.5003
2	0.5111	0.5160	0.4946	0.4992	0.5001	0.5001	0.5000	0.5000	0.5000
3	0.4965	0.5022	0.5003	0.5000	0.5000	0.5000			
4	0.4992	0.5001	0.5001						
5	0.4999	0.5000	0.5000						
6	0.5000	0.5000	0.5000						

Tabel 2.
Result of comparison spectral radius, iteration number

Method	Gauss -Seidel	RGS	IRGS
Spectral Radius	0.1291	0.0167	0.0021
Rate convergence	2.0472	4.0923	6.1658,
Iteration Number	6	3	2

Table 3.
Spectral radius for SOR, RSOR, and IRSOR

φ	ρ_{SOR}	ρ RefineSOR	ρ Improve SOR
0.2	0.8542	0.7296	0.6933
0.4	0.6947	0.4827	0.3353
0.6	0.6016	0.3620	0.2178
0.8	0.7767	0.6035	0.4686
1	0.1291	0.0167	0.0021
1.2	0.2535	0.0643	0.0163
1.4	0.4440	0.1972	0.0875
1.6	0.6328	0.4004	0.2534
1.8	0.8182	0.6695	0.5477
1.9	0.9096	0.8272	0.7525

Table 4.
Rate convergence for SOR, RSOR, and IRSOR

φ	SOR	RSOR	IRSOR
0.2	0.1576	0.3153	0.3663
0.4	0.3643	0.7284	1.0927
0.6	0.5082	1.0161	1.5242
0.8	0.2527	0.5050	0.7580
1	2.0472	4.0923	6.16558
1.2	1.3724	2.7442	4.1166
1.4	0.8119	1.6235	2.4361
1.6	0.4576	0.9153	1.3728
1.8	0.2006	0.4012	0.6020
1.9	0.0948	0.1897	0.2844

3. Result and Discussion

In section 2, the gauss-seidel and SOR iterative refinement method formulations and numerical case is presented to calculate their performance. Using MATLAB gives the exact solution is 0.5. Table 1 shows the solution of Gauss-Seidel method, refinement, and iterative refinement numerical results. The results show that the Gauss seidel method requires 6 iterations to find the exact value. The refinement Gauss-seidel formulation yields from derivate form of ordinary Gauss-seidel, and particularly for iterative refinement Gauss-seidel. The solution obtain for refinement Gauss-seidel is 5 iteration, and Iterative refinement is 2 iteration. This means that the performance of iterative refinement method is faster than the two other methods. This is shown, from the number of iterations. In terms of spectral radius, Gauss-seidel to its second derivative has decreased. However, contrary to the rate of convergence which has increased as shown in table 2. This result is consistent with research by [21], they solved a problem with a reduced number of iterations and faster rate of convergence than previous methods. The SOR method is an extension of the Gauss-seidell method by adding a relaxation parameter. In this case, the parameter values tested were [0.2,1.9]. Table. 3 presents the results of spectral radius values using relaxation values (φ) for SOR, SOR refinement, and improved SOR refinement. In each SOR method, the spectral radius distance value is getting smaller, especially when the relaxation value (φ) is given 1, which is 0.0021 in the spectral radius of the improved refinement SOR method. A comparison of the spectral radius graph is shown in Figure 1.

Result Spectral radius for SOR , RSOR, and IRSOR

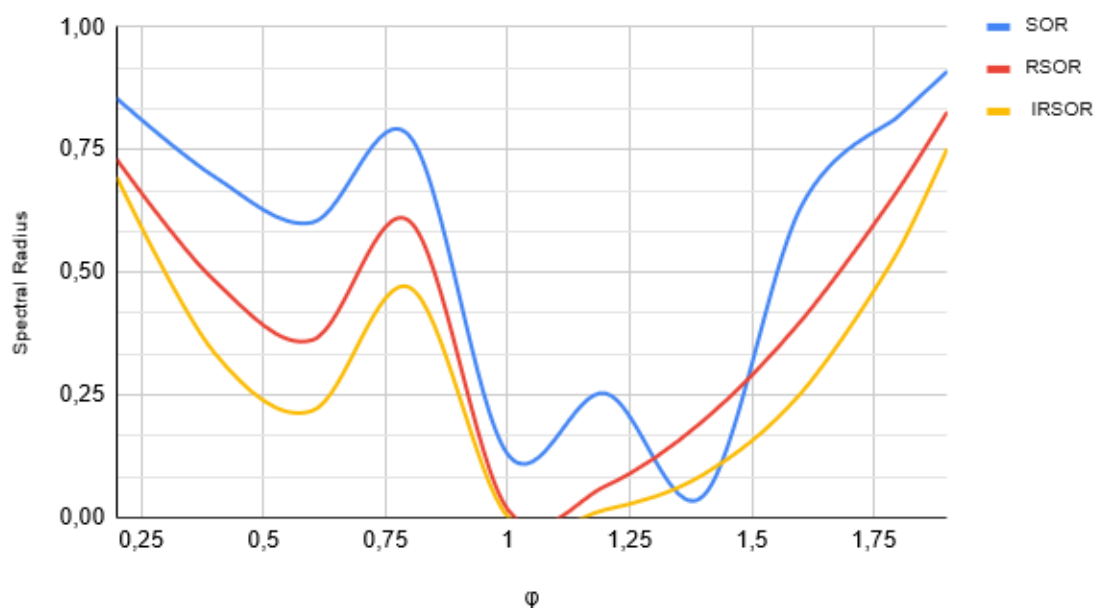


Figure 1. Comparison spectral radius SOR,RSOR,and IRSOR

A comparison of the number of iterations for the SOR, RSOR, and IRSOR methods is shown in figure 2. The optimum number of iterations in this case is located at a relaxation value (φ) of 1. The number of iterations increases if the relaxation value (φ) gets closer to 2. However, the IRSOR method provides the best performance by reducing the number of iterations compared to the performance of RSOR and SOR.

Comparison The number Iteration SOR, RSOR, IRSOR

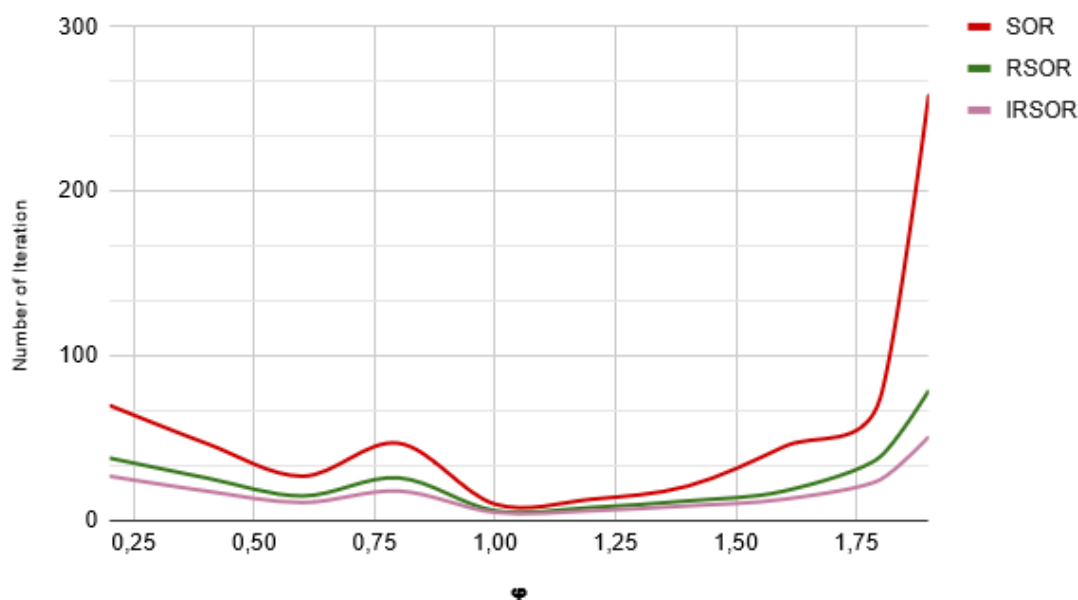


Figure 2. Comparison Number Iteration of SOR,RSOR,and IRSOR

In terms of convergence rate, the optimum value is given at relaxation 1 in IRSOR of 6.16558 as shown in table 4. This shows that the IRSOR method gives the best performance in terms of convergence rate and spectral radius. However, the relaxation value (ϕ) of that exceeded 1 has an impact on the number of iterations increasing to approach the exact value. This research is limited to discussing performance at the convergence rate and spectral radius. Future research is expected to be studied with a large rank that affects a more complex sparse matrices. The relaxation value that gives the optimum approach also gives a larger error value, so that in the next research it is studied with a stochastic approach to get the optimum solution.

4. Conclusion

In this paper, Gauss-seidel and successive overrelaxation (SOR) with their refinement and improve refinement for solving sparse matrices is studied. Sufficient conditions for convergence are given and numerical experiments are considered to show the efficiency of the method. Sufficient conditions for convergence and spectral radius are given and numerical experiments are considered to show the efficiency of these methods. The main findings in this study, that Gauss seidel improvement refinement gives optimum spectral radius and convergence rate. Similarly, the SOR improved refinement method gives. Considering their performance, using parameters such as time to converge, number of iterations required to converge and spectral radius level of accuracy. However, SOR works with relaxation values so that it greatly affects the convergence rate and spectral radius results if given greater than 1. This research is limited to discussing performance at the convergence rate and spectral radius. Future research is expected to be studied with a large rank that affects a more complex sparse matrices. The relaxation value that gives the optimum approach also gives a larger error value, so that in the next research it is studied with a stochastic approach to get the optimum solution.

References

- [1] A. K. Maurya, A. S. Yadav, A. Kumar, and K. K. Kushwaha, "An Overview on Numerical Solution of System," *Int. J. Innov. Res. Sci. Eng. Technol.*, vol. 6, no. 5, pp. 9476–9481, 2017, doi:

- 10.15680/IJIRSET.2017.0605290.
- [2] M. Kryshchuk and J. Lavendels, "Iterative Method for Solving a System of Linear Equations," *Procedia Comput. Sci.*, vol. 104, no. December 2016, pp. 133–137, 2016, doi: 10.1016/j.procs.2017.01.085.
 - [3] S. Siswanti, S. Syamsudhuha, and S. Putra, "Metode Iterasi Aor untuk Sistem Persamaan Linear Prekondisi," *JOM FMIPA*, vol. 2, no. 1, pp. 387–396, 2015, [Online]. Available: <https://www.neliti.com/publications/188697/metode-iterasi-jacobi-dan-gauss-seidel-prekondisi-untuk-menyelesaikan-sistem-per%00Ahttps://www.neliti.com/publications/186001/metode-iterasi-aor-untuk-sistem-persamaan-linear-prekondisi>
 - [4] D. Yan, T. Wu, Y. Liu, and Y. Gao, "An efficient sparse-dense matrix multiplication on a multicore system," in *International Conference on Communication Technology Proceedings, ICCT*, 2017, vol. 17, pp. 1880–1883, doi: 10.1109/ICCT.2017.8359956.
 - [5] H. Alyahya, R. Mehmood, and I. Katib, *Parallel iterative solution of large sparse linear equation systems on the intel mic architecture*. 2020. doi: 10.1007/978-3-030-13705-2_16.
 - [6] J. D. Booth and G. S. Bolet, "Neural Acceleration of Graph Based Utility Functions for Sparse Matrices," *IEEE Access*, vol. 11, no. February, pp. 31619–31635, 2023, doi: 10.1109/ACCESS.2023.3262453.
 - [7] S. Masanawa and H. Abubakar, "Sparse Matrix Approach in Neural Networks for Effective Medical Data Sets Classifications," *J. Basic Appl. Res. Biomed.*, vol. 6, no. 2, pp. 90–97, 2020, doi: 10.5152/jbarbiomed.v6i2.113.
 - [8] S. Chen, J. Fang, C. Xu, and Z. Wang, "Adaptive Hybrid Storage Format for Sparse Matrix–Vector Multiplication on Multi-Core SIMD CPUs," *Appl. Sci.*, vol. 12, no. 19, 2022, doi: 10.3390/app12199812.
 - [9] J. Wen, Y. Y. Zhang, and S. S. T. Yau, "k-mer Sparse matrix model for genetic sequence and its applications in sequence comparison," *J. Theor. Biol.*, vol. 363, pp. 145–150, 2014, doi: 10.1016/j.jtbi.2014.08.028.
 - [10] S. Mohr, W. Dawson, M. Wagner, D. Caliste, T. Nakajima, and L. Genovese, "Efficient Computation of Sparse Matrix Functions for Large-Scale Electronic Structure Calculations: The CheSS Library," *J. Chem. Theory Comput.*, vol. 13, no. 10, pp. 4684–4698, Oct. 2017, doi: 10.1021/acs.jctc.7b00348.
 - [11] R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations*. Seattle Washington: SIAM, 2007. doi: 10.1137/1.9780898717839.
 - [12] Y. Saad, "Iterative Methods for Sparse Linear Systems," *Iterative Methods Sparse Linear Syst.*, 2003, doi: 10.1137/1.9780898718003.
 - [13] U. Borštnik, J. VandeVondele, V. Weber, and J. Hutter, "Sparse matrix multiplication: The distributed block-compressed sparse row library," *Parallel Comput.*, vol. 40, p. 47, 2014.
 - [14] A. K. Maulana and F. Fitriyani, "Evaluasi Kinerja Sparse Matrix-vector Multiplication Menggunakan Format Penyimpanan Csr Dan Bcsr Pada Mpi," in *eProceedings of Engineering*, 2017, vol. 4, no. 1, pp. 1301–1309.
 - [15] W. Yang, K. Li, and K. Li, "A parallel computing method using blocked format with optimal partitioning for SpMV on GPU," *J. Comput. Syst. Sci.*, vol. 92, pp. 152–170, 2018, doi: 10.1016/j.jcss.2017.09.010.
 - [16] T. K. Enyew, G. Awgichew, E. Haile, and G. D. Abie, "Second-refinement of Gauss-Seidel iterative method for solving linear system of equations," *Ethiop. J. Sci. Technol.*, vol. 13, no. 1, pp. 1–15, 2020, doi: 10.4314/ejst.v13i1.1.
 - [17] and G. C. F. P. P. Koester, S. Ranka, "A Parallel Gauss-Seidel Algorithm for Sparse Power System Matrices," *IEEE Trans. Power Syst.*, pp. 184–193, 1994.
 - [18] Z. Virag, "Improve Symmetric Gauss-Seidel Method For Solving Sparse Linier Systems Appear In CFD," in *International Congress of Croatia Society of Mechanics*, 2006, pp. 21–23.
 - [19] R. Abdullahi and R. Muhammad, "Refinement Of Preconditioned Overrelaxation Algorithm For Solution of The Linear Algebraic System $Ax=b$," *Sci. World J.*, vol. 16, no. 3, pp. 226–231, 2021.
 - [20] Y. Saad, *Parallel iterative methods for sparse linear systems*, vol. 8, no. C. Elsevier Masson SAS, 2001. doi: 10.1016/S1570-579X(01)80025-2.
 - [21] G. Dessalew, T. Kebede, G. Awgichew, and A. Walelign, "Generalized Refinement of Gauss-Seidel Method for Consistently Ordered 2-Cyclic Matrices," *Abstr. Appl. Anal.*, vol. 2021, 2021, doi: 10.1155/2021/8343207.
 - [22] D. M. Young, "Chapter 4 - CONVERGENCE OF THE BASIC ITERATIVE METHODS," in *Iterative Solution of Large Linear Systems*, D. M. Young, Ed. Academic Press, 1971, pp. 106–139. doi: <https://doi.org/10.1016/B978-0-12-773050-9.50012-7>.