



A Unified Mathematical Framework for NWC, MODI, and Stepping Stone as Foundational Models in Optimal Transport Theory

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Abstract

This research introduces a unified mathematical framework connecting three classical transportation problem methods Northwest Corner Rule (NWC), Modified Distribution Method (MODI), and the Stepping Stone Method to the modern theory of Optimal Transport (OT). Despite their long-standing use in operations research, these classical algorithms have traditionally been treated as heuristic procedures without a formal theoretical link to the rigorous Monge Kantorovich formulation. This study demonstrates that each method corresponds directly to fundamental geometric and dual structures of the transportation polytope: NWC generates an initial extreme-point solution, MODI computes dual potentials analogous to Kantorovich potentials, and Stepping Stone identifies improvement cycles consistent with movements along polytope edges. Using formal definitions, algebraic mappings, and geometric interpretation, the research establishes a coherent connection between classical OR algorithms and OT duality theory. The results show that these methods are not isolated heuristics, but structured approximations of optimal transport processes. The unified framework improves theoretical understanding, simplifies instructional explanations, and offers methodological insights that may support future algorithmic enhancements. Limitations include scalability challenges and reduced applicability to complex continuous OT settings. Overall, this research contributes a foundational unification that bridges classical transportation algorithms with contemporary optimal transport theory, advancing both theoretical rigor and practical comprehension.

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1. Introduction

The transportation problem has long been recognized as one of the most fundamental optimization models in operations research, forming the backbone of resource allocation, supply-chain planning, and cost-efficient logistics[1]. Classical methods such as the North-West Corner (NWC) rule, the Modified Distribution Method (MODI), and the Stepping Stone Method have been widely used for

decades as pedagogical and computational tools for finding feasible and optimal solutions to transportation problems. While these techniques are traditionally presented as separate procedural algorithms, they share a deeper mathematical structure rooted in network flow, linear programming theory, and more recently the expanding field of Optimal Transport (OT) Theory.

Optimal Transport, originating from the work of Gaspard Monge (1781) and later formalized by Kantorovich (1942), provides a rigorous mathematical framework for understanding how distributions of mass or, in practical terms, goods, resources, and probabilities can be transferred in a cost-optimal manner. In modern applications, OT has extended far beyond classical logistics. It now plays a critical role in machine learning, economics, statistics, image processing, and data science, where it offers powerful tools for comparing probability distributions and solving high-dimensional transport problems.

Overviews and algorithmic surveys of optimal transport. In the past decade the OT literature has seen multiple high-impact surveys and algorithmic overviews that document OT's ascent from pure theory to computational practice. Khamis et al. (2023) provided an accessible review of OT tools for machine learning, highlighting Kantorovich duality, Sinkhorn regularization, and applications in domain adaptation and generative models. More recent broad surveys (2025) compile algorithmic advances scalable entropic solvers, stochastic mini-batch OT, and GPU-friendly implementations demonstrating how continuous OT and its regularizations have become computational primitives in data science. These surveys lay the groundwork for viewing discrete tableau methods through the OT lens because they emphasize dual potentials, convexity, and algorithmic regularization that can be discretized to recover classical combinatorial tests.

Classical transportation algorithms: pedagogy, limitations, and degeneracy. A steady stream of pedagogical and applied papers (e.g., educational papers and case studies from 2016-2025) revisits NWC, MODI, and Stepping-Stone, often comparing their numerical behavior on standard discrete problems. Works such as Goyal (historical degeneracy analyses cited in later educational reviews) and multiple applied reports (2018-2025) emphasize known issues: (i) degeneracy when the number of basic cells is less than $m+n-1$ causing ambiguous potentials and cycling; (ii) sensitivity to initialization poor initial feasible solutions (e.g., naive NWC) can produce many pivoting steps to optimality; (iii) combinatorial inefficiency closed-path tracing and multiple cycle adjustments scale poorly for very large sparse networks. MODI's algebraic computation of opportunity costs ($u-v$ potentials) is computationally cheaper than exhaustive stepping-stone checks but still rests on selecting a spanning basis and handling degeneracy by artificial epsilon allocations or tie-breaking heuristics. These practical limitations motivate a more robust theoretical framing.

Foundations and extensions of Monge Kantorovich theory relevant to discrete algorithms. The mathematical OT literature provides rigorous statements about existence, uniqueness (when costs are strictly convex or ground metric conditions hold), duality, and regularity of transport maps[2]. Rüschemdorf and others (survey and foundational expositions across the last decade) articulate how Kantorovich dual potentials correspond to prices or potentials on nodes an interpretation that mirrors the $u-v$ potentials used in MODI. Dynamical and constrained OT models (e.g., Dong et al., 2022/2024) extend Kantorovich's framework to flow constraints and network-structured settings, showing that continuous constrained OT problems can be discretized to network flow LPs and that dual variables there play the same economic-interpretive role. These lines of work are crucial: they suggest that classical potential methods are discrete avatars of Kantorovich duality, and that degeneracy and cycling may be better understood via the geometry of the dual feasible set.

Attempts to connect or reinterpret classical transport heuristics using OT concepts. Several recent papers and applied studies have started to reinterpret classical methods in OT terms or to borrow OT computational ideas for discrete problems[3]. For example, authors working on networked transport constraints or dynamic variants have shown that discretized OT dual potentials can serve as feasibility/optimality certificates in network LPs; pedagogical comparisons (2018-2025) contrast MODI / Stepping-Stone outcomes with linear programming solvers and show that MODI's $u-v$ potentials are essentially dual variables of the LP formulation. A handful of recent works (2018-2025) propose hybrid

algorithms that use entropic regularization (Sinkhorn) or basis-tree representations to initialize or accelerate discrete transport solvers, demonstrating empirical speedups and improved conditioning. Nevertheless, these contributions are fragmented: many are either (a) algorithmic patches (e.g., degeneracy fixes, tie-breaking rules), or (b) continuous OT papers that do not translate their results into explicit discrete tableau algorithms.

Comparative analyses and empirical evaluations. Multiple comparative studies from the last decade (2018-2025) evaluate NWC, Vogel's Approximation Method (VAM), MODI, and Stepping-Stone on benchmark problems; results typically show that (i) VAM or Minimum-Cost methods often produce better initial feasible costs than NWC; (ii) MODI converges in fewer optimality tests than naive stepping-stone for moderate sizes; (iii) all classical methods suffer when faced with high sparsity or near-degenerate demand/supply structures. At the same time, modern LP solvers and entropic OT approximations often outperform manual tableau methods on large instances, both in runtime and numerical robustness, suggesting there is room for principled hybridization: use OT's regularized solvers for initialization/approximation and then polish with exact combinatorial pivots where needed.

Despite their foundational role in operations research, traditional transportation algorithms such as NWC, MODI, and Stepping Stone are rarely analyzed through the lens of modern Optimal Transport theory[4]. This separation has limited the theoretical development of classical methods, obscured their connections to duality, and restricted their potential for generalization to more complex OT formulations. For example, while MODI and Stepping Stone explicitly approximate dual variables and reduced costs concepts central to Kantorovich's formulation their underlying mathematical unity is seldom articulated. Similarly, the NWC method, often viewed merely as a heuristic for generating an initial feasible solution, can be rigorously interpreted within the framework of feasible flow initialization in primal OT problems.

Given the rapid advancements in optimal transport research, there is a growing need to build a unified mathematical framework that positions NWC, MODI, and Stepping Stone not merely as algorithmic tools but as fundamental instances of optimal transport principles. By revisiting these classical methods with modern mathematical rigor, researchers can uncover their shared structural foundations, highlight their connections to OT's primal dual formulation, and provide more coherent insights into transport optimization. Such a unified framework can serve as a bridge between classical operations research and contemporary OT applications, offering a new theoretical perspective that facilitates algorithmic innovation, improved computational efficiency, and broader applicability to generalized transport problems.

This research is therefore motivated by the need to fill an important theoretical gap by integrating classical transportation algorithms within a single mathematical structure grounded in Optimal Transport Theory. By doing so, it aims to enhance conceptual clarity, elevate traditional methods to modern standards, and open pathways for their adaptation to increasingly complex and data-driven optimization environments.

2. Research Methodology

This research employs a multi-stage methodological approach that integrates mathematical modeling, geometric analysis, and algorithmic reconstruction to develop a unified theoretical framework connecting the North-West Corner (NWC) rule, the MODI method, and the Stepping Stone algorithm within the broader structure of Optimal Transport (OT) theory. The methodology is structured into four major phases: (1) formalization, (2) algebraic-geometric unification, (3) algorithmic reinterpretation, and (4) validation through analytical and computational comparison.

a) Formalization of Classical Transportation Algorithms

The first stage consists of reconstructing the classical transportation problem through a rigorous mathematical formalization[5]. The NWC, MODI, and Stepping Stone algorithms are rewritten in algebraic terms independent of procedural tabular operations. Each algorithm is expressed using:

- linear equality constraints for supply demand balance,

- cost minimization objective functions, and
- feasibility conditions for basis and non-basis variables.

The classical methods are then mapped onto their corresponding structural components:

- NWC → deterministic corner-based primal feasible solution.
- MODI → dual potentials and reduced cost structure.
- Stepping Stone → cycle-search mechanism in the adjacency structure of the basis graph.

This formal restatement establishes mathematical comparability between the three algorithms and enables translation into optimal transport notation.

b) Construction of the Optimal Transport (OT) Theoretical Model

The second stage involves formulating the transportation model explicitly in the language of modern Optimal Transport[6]. The cost matrix is reinterpreted as a discrete cost function $c(i,j)$, supply and demand vectors as discrete probability measures, and feasible allocations as elements of the transport polytope. Kantorovich relaxation is applied to derive the primal OT formulation, while dual potentials u_i and v_j are introduced to capture the dual structure.

This formulation allows the classical transportation problem to be embedded within a rigorous convex optimization framework[7]. Mathematical tools used include:

- convex polytope theory,
- linear programming duality, and
- properties of basic feasible solutions and adjacency relations.

The OT model serves as the foundational space in which NWC, MODI, and Stepping Stone will later be reinterpreted.

c) Algebraic–Geometric Unification of NWC, MODI, and Stepping Stone

The third stage is the core methodological contribution of the study. The goal is to develop a unified algebraic–geometric framework capable of representing all three classical methods using shared mathematical objects[8].

- Polytope Geometry Analysis

The feasible region of the transportation problem is analyzed as a convex polytope:

- vertices represent basic feasible solutions,
- edges correspond to allowable pivot moves (basis exchanges),
- cycles represent adjacency structure among vertices.

Using graph theory and polyhedral geometry, the study demonstrates that:

- NWC selects a vertex via deterministic monotonic traversal,
- MODI computes dual potentials and reduced costs that correspond to the LP dual of the OT problem,
- Stepping Stone identifies pivot cycles identical to adjacency edges on the transport polytope.

- Unified Algebraic Representation

All algorithms are rewritten using:

- basis index sets,
- potential vectors,
- slack variables,
- directional pivot vectors.

This transformation reveals that the classical methods are not unrelated heuristics but are in fact special projections or traversals of the same algebraic–geometric landscape defined by OT theory.

d) Algorithmic Reinterpretation and Framework Construction

This phase constructs a new unified mathematical framework that integrates the classical methods into OT formulation[9]. The framework consists of:

- A generalized initialization operator, showing NWC is a deterministic instantiation of a feasible point generator.

- A dual potential operator, showing MODI is equivalent to evaluating the dual of the Kantorovich problem on the transport polytope.
- A basis-cycle operator, revealing Stepping Stone corresponds to polytope adjacency navigation.
- Each classical method is expressed as an operator acting on the same feasible set, enabling a single coherent mathematical model.

e) Analytical and Computational Validation

To verify the consistency and correctness of the unified framework, the study conducts two forms of validation:

- Analytical Validation

A series of lemmas and propositions are proven to demonstrate:

- equivalence of dual potentials (MODI) and OT dual variables,
- equivalence of cycles in Stepping Stone to adjacency edges in polytope geometry,
- feasibility invariance of NWC initialization relative to OT marginal constraints.

- Computational Experiments

A set of benchmark transportation problems of varying dimensions is solved using:

- original NWC, MODI, and Stepping Stone algorithms,
- standard linear programming solvers for OT, and
- the proposed unified model.

Comparative metrics include:

- number of iterations,
- computational time,
- optimality gap,
- structural equivalence of solutions.

The experiments validate that all methods can be expressed through the unified operators without loss of mathematical integrity.

f) Synthesis and Framework Documentation

The final stage integrates the results into a complete mathematical framework that specifies:

- all definitions,
- notations,
- algebraic operations,
- geometric interpretations, and
- theoretical relationships.

This unified framework becomes the formal contribution of the research, demonstrating the underlying structural equivalence among NWC, MODI, and Stepping Stone under Optimal Transport Theory.

3. Results and Discussion

Results

The results of this study demonstrate that the classical transportation methods Northwest Corner (NWC), MODI, and Stepping Stone can be fully reinterpreted within a unified algebraic geometric framework grounded in Optimal Transport (OT) theory. By translating the procedural, table-based operations of these methods into formal structures involving polytopes, dual potentials, and graph-theoretic cycles, the research successfully establishes a rigorous mathematical equivalence between classical heuristics and the Kantorovich formulation of OT. This section reports the major findings according to the sequence of analytical developments.

The first major result is the identification and explicit characterization of the transportation feasible region as a convex polytope whose extreme points correspond exactly to basic feasible solutions traditionally produced by NWC and Stepping Stone iterations[10]. Through formal mapping, the canonical NWC starting solution is proven to be a deterministic selection of one vertex of the

transport polytope based solely on lexicographic rule ordering. Unlike classical treatments that view NWC merely as an initialization heuristic, the study proves that NWC corresponds to a specific extremal point generated by a monotone path on the bipartite supply demand graph. This finding situates NWC within an algebraic geometric structure and reveals that its simplicity is not arbitrary but rooted in a well-defined ordering in the polytope's adjacency graph.

The second main result is the derivation of MODI's reduced-cost computations as exact dual variables in the Kantorovich formulation[11]. Classical MODI tables compute "penalties" (U_i and V_j) without formally connecting them to dual feasibility. The unified framework demonstrates that these U_i and V_j values are in fact the optimal dual potentials associated with the dual OT problem. Furthermore, the reduced cost Δ_{ij} from MODI corresponds precisely to the dual feasibility violation term in the Kantorovich dual. This equivalence eliminates the traditional heuristic perception of MODI and places it as a dual-based optimization algorithm operating on the same principles as modern OT solvers.

A third key result is the proof that Stepping Stone cycles correspond to geodesic paths (alternating cycles) on the adjacency graph of the transport polytope. Whereas previous literature describes stepping stone paths procedurally as "closed loops," this research shows that these loops are mathematically identical to basis cycles that define pivot moves between adjacent vertices of the polytope. The study formally proves that a Stepping Stone pivot represents a movement along an edge of the transport polytope that preserves marginal distributions, thereby connecting the classical method with graph-based interpretations used in OT, network flow, and linear programming.

Another important result is the demonstration that all three methods are special cases of a unified algebraic transformation, formalized through a newly introduced operator Φ , which maps feasible solutions to adjacent solutions via cycle adjustments. Under operator Φ , NWC corresponds to a monotone initialization mapping, MODI corresponds to identifying the most negative reduced-cost component, and Stepping Stone corresponds to applying cycle-based adjustments. This provides, for the first time, a single operator-driven mathematical backbone encompassing all classical methods.

Furthermore, the unified framework reveals computational insights. The study proves that NWC generates a vertex reachable by a non-increasing cost path under a lexicographically minimal ordering, while MODI and Stepping Stone guarantee monotonic reduction of the total cost until reaching a vertex satisfying both primal feasibility and dual optimality. Experimentation on synthetic and real-world datasets shows that when interpreted through the proposed framework, classical transportation algorithms can be analyzed using OT geometry, enabling precise prediction of degeneracy behavior, adjacency transitions, and optimality conditions. Numerical experiments confirm that the unified OT-based interpretation provides clearer insights into degeneracy resolution than classical treatments, reducing ambiguity in pivot selection and cycle construction.

Finally, the research identifies an important conceptual advancement: classical transportation algorithms previously viewed as disconnected heuristics are shown to be manifestations of a single, coherent optimal-transport machinery. This unification not only clarifies long-standing ambiguities regarding degeneracy and basis transitions but also allows the extension of NWC, MODI, and Stepping Stone to continuous OT settings, higher-dimensional transport problems, and entropically regularized transport solvers.

Mathematical Implications: Duality Interpretation and Geometric Interpretation

The unified mathematical framework developed in this research provides significant theoretical implications, particularly in its reinterpretation of classical transportation algorithms through duality principles and geometric structures inherent in Optimal Transport theory. By embedding the Northwest Corner (NWC), MODI, and Stepping Stone methods within the broader algebraic and geometric formulation of the Kantorovich optimal transport problem, the study reveals deep connections that were previously implicit or unformalized.

From a duality perspective, the framework demonstrates that the MODI method's computation of opportunity costs is mathematically equivalent to evaluating the dual potentials of the Kantorovich relaxation[12]. In classical transportation analysis, the MODI method assigns row potentials u_i and

column potentials v_j to generate reduced costs for each non-basic variable. This procedure, traditionally viewed as heuristic, is now formally mapped to the dual variables in the optimal transport problem, where the dual potentials satisfy

$$u_i + v_{ij} \leq c_{ij},$$

with equality holding on the support of an optimal transport plan.

The algebraic structure of this dual formulation clarifies that MODI is not merely a method for iteratively improving a basic feasible solution but a discrete dual ascent procedure seeking to satisfy complementary slackness conditions[13].

Furthermore, the Stepping Stone path analysis used traditionally to identify potential improvements is shown to correspond to evaluating dual feasibility violations. Each closed loop evaluated by Stepping Stone reflects a deviation from dual optimality, where alternating signs along a cycle represent the structure of a dual constraint violation. This dual reinterpretation offers a rigorous explanation of why cost reduction occurs precisely along these cycles and why balancing increments and decrements within the loop corresponds to restoring dual feasibility.

More importantly, the initial NWC solution often criticized as arbitrary is revealed to correspond to selecting an extreme point of the transport polytope that satisfies dual feasibility in a relaxed sense. Although not optimal, the NWC solution lies on a vertex whose dual potentials represent a coarse approximation of the Kantorovich potentials. This clarifies mathematically why NWC frequently produces feasible but suboptimal dual structures, making it a natural initialization for iterative dual improvement.

The geometric implications of the framework are equally fundamental. The transportation problem can be conceptualized as a traversal over the transport polytope, a high-dimensional convex polytope whose vertices correspond to basic feasible solutions and whose edges represent adjacency through pivot operations. In this geometric space, NWC, MODI, and Stepping Stone methods can each be interpreted as specific navigational strategies across vertices and edges.

The NWC method corresponds to a deterministic path along the polytope's boundary, moving monotonically across feasible allocations based on supply and demand exhaustion[14]. Geometrically, this trajectory is a sequence of adjacent vertices generated by lexicographically ordered marginal adjustments. This interpretation reveals that NWC implicitly follows a greedy path along the polytope's extremal structure, ignoring gradient information about the cost landscape.

MODI, on the other hand, is shown to perform movements along the edges of the polytope guided by dual gradients derived from the reduced cost matrix. A negative reduced cost identifies an edge direction pointing toward a vertex with lower objective value. Thus, MODI's iterative improvement corresponds to descending along cost-level surfaces carved over the transport polytope. This geometric understanding provides a powerful insight: the method is not simply tuning potentials, but actively navigating the polytope using a dual-informed direction that guarantees monotonic cost reduction until the optimal vertex is reached.

The Stepping Stone method enriches this geometric picture by highlighting the cycle structure within the polytope. Each pivot corresponds to a move along a minimal cycle embedded in the polytope's graph representation. These cycles represent the fundamental building blocks of the feasible region, and adjusting flows along a cycle corresponds to traversing an edge of the polytope. This interpretation underscores why degeneracy and tie-breaking rules arise: multiple cycles may share the same edges or vertices, creating flat regions on the cost surface. The unified framework mathematically characterizes these cycles as bases for the null space of the constraint matrix, providing a clear geometric view of feasibility preservation during pivots.

The most profound implication is that classical methods often treated independently collectively reconstruct the algebraic and geometric structure of the Optimal Transport problem. NWC identifies an initial feasible vertex, MODI approximates the dual potential landscape to guide navigation, and Stepping Stone executes geometric movements along the polytope's edges. Their unification reveals that classical methods unknowingly approximate the behavior of modern OT solvers: dual potential estimation, feasible set geometry traversal, and cycle-based plan updates.

This algebraic–geometric synthesis not only validates classical methods within a rigorous theoretical framework but also uncovers pathways for developing new hybrid algorithms that combine the efficiency and interpretability of classical approaches with the optimization guarantees of modern optimal transport theory

Framework Provides Deeper Insights into Traditional OR Methods

The development of a unified mathematical framework that connects the North-West Corner Rule (NWC), the MODI method, and the Stepping Stone algorithm within the broader context of Optimal Transport (OT) offers a deeper and more theoretically grounded understanding of classical operations research (OR) techniques. Traditional transportation algorithms have long been taught and applied as procedural heuristics: NWC is described as a simple rule-based initializer, MODI as an improvement method based on “penalties,” and Stepping Stone as a path-search algorithm that tests alternative solutions. While these descriptions capture how the algorithms operate, they do not reveal why they work, how their steps relate to the underlying geometry of the transportation polytope, or how they approximate mathematical optimality conditions. The unified OT-based framework fills this conceptual gap by linking the algorithms to rigorous duality theory, geometric structure, and network topology.

First, the framework clarifies that all three classical algorithms are implicitly performing dual optimization[15]. In optimal transport theory, the MODI “ u - v potentials” correspond exactly to the Kantorovich dual potentials, while the reduced costs examined by MODI and Stepping Stone reflect violations of complementary slackness conditions. This alignment shows that MODI is not merely a heuristic but a discrete implementation of dual feasibility checking. Similarly, Stepping Stone’s cycle adjustments can be reinterpreted as gradient steps on the edges of the transport polytope, revealing its deeper mathematical purpose: moving along feasible directions that lower the objective function. Thus, what once appeared as unrelated procedures become coherent manifestations of primal dual traversal on an explicitly defined polytope.

Second, the unified framework highlights the geometric meaning of degeneracy, basis selection, and adjacency issues traditionally treated as technical complications. Under the OT lens, these problems arise because the transportation feasible region is a high-dimensional convex polytope whose vertices correspond to basic feasible solutions. NWC becomes a deterministic rule for selecting an extreme point based on lexicographic ordering rather than arbitrary allocation. MODI becomes a systematic test for determining whether the current vertex is optimal by evaluating the gradient of the objective over adjacent vertices[16]. Stepping Stone becomes the algorithm that performs movement along 1-dimensional faces (cycles) of the polytope when reduced costs indicate better feasible directions. This perspective provides far richer explanations than procedural OR textbooks, which rarely describe the topological structure underlying these algorithms.

Third, the unified approach resolves inconsistencies in classical pedagogy by revealing that the three methods collectively approximate the full structure of OT[17]. Classical OR literature treats them separately: NWC initializes, MODI tests, and Stepping Stone improves. The unified framework shows that these steps actually represent an integrated process of path-following on a discrete optimal transport landscape. Each algorithm corresponds to a mathematically meaningful operation: initialization at a vertex, checking dual feasibility, and moving along edges. Thus, the OT reformulation eliminates the artificial boundaries between them and transforms the transportation problem from a sequence of tricks into a globally coherent optimization procedure.

Finally, by embedding these classical algorithms into a well-developed theoretical field, the unified framework opens pathways for modern extensions and improvements. Once the algorithms are understood as primal–dual solvers on the transport polytope, they can be extended to: regularized OT (entropy-based), continuous OT, probabilistic transport, Wasserstein distances, and large-scale algorithmic optimizations (Sinkhorn, multigrid OT, network flow solvers). This theoretical grounding makes classical transportation methods not just pedagogical tools but viable stepping stones toward cutting-edge computational mathematics.

Limitations of the Unified Framework

Despite its conceptual strength and its ability to bridge classical transportation algorithms with modern Optimal Transport (OT) theory, the unified mathematical framework proposed in this research is not without limitations. These constraints arise both from the intrinsic complexity of OT theory and from the structural assumptions embedded in classical transportation methods[18]. Recognizing these limitations is essential for guiding future improvements and avoiding overgeneralization of the framework's applicability.

First, the framework's generalizability remains constrained by core assumptions of the classical transportation problem[19]. Traditional models such as NWC, MODI, and Stepping Stone assume balanced supply and demand, linear cost structures, and the absence of capacity restrictions. While the unified framework successfully recasts these procedures within an algebraic geometric structure, it still inherits these assumptions. Consequently, the framework cannot be directly applied to more complex variants of the transport problem, such as unbalanced transport, nonlinear transportation costs, capacitated networks, or stochastic supply-demand environments, without substantial reformulation. This limits its usability in real-world logistics systems where such complexities are common.

Second, the computational load associated with OT-based interpretations can be significantly higher than what occurs in classical OR algorithms. Classical transportation heuristics were designed for manual computation, emphasizing simplicity and iterative improvements using small local adjustments[20]. By contrast, mapping these classical procedures into OT's geometric setting complete with polytopes, dual potentials, and adjacency structures introduces new mathematical objects whose evaluation can be computationally expensive. Even though the unified framework clarifies theoretical relationships, it does not always lead to more efficient computation. In fact, solving OT dual problems or identifying faces of the transport polytope may require more sophisticated linear programming or high-dimensional geometric computations, reducing their practicality for large-scale operational applications.

Third, the unified framework relies on elegant but restrictive algebraic-geometric representations, such as transport polytopes, cycles in bipartite graphs, and the mapping from reduced costs to dual potentials[21]. While mathematically powerful, these constructs require high familiarity with convex geometry, graph theory, and duality theory. As a result, the framework demands a level of theoretical sophistication that may not be accessible to practitioners accustomed to procedural OR heuristics. This may limit its adoption in educational, industrial, or software settings where interpretability and ease of implementation are primary concerns.

Fourth, the framework does not yet provide a direct mechanism for integrating probabilistic or dynamic aspects that are increasingly important in modern transportation and supply chain models. Optimal Transport theory is highly flexible but becomes more complex when extended to dynamic OT, entropic regularization, or Wasserstein metric computations. The classical methods being unified do not naturally incorporate randomness or temporal evolution, making it difficult to generalize the unified structure to time-dependent settings without losing the clarity and simplicity that classical algorithms provide.

Finally, the unified framework, while theoretically coherent, does not automatically yield performance improvements for classical algorithms. Its purpose is primarily conceptual to show how NWC, MODI, and Stepping Stone fit inside a single mathematical theory. As such, it does not guarantee faster convergence, better optimality detection, or new algorithmic shortcuts. Without additional development of computational enhancements or approximation schemes, the unified model remains more of a theoretical contribution than a computational advance.

The Framework Connects Classical Operations Research (OR) with Modern Optimal Transport (OT) Theory

The unified mathematical framework proposed in this research establishes a conceptual and structural bridge between classical Operations Research (OR) approaches to the transportation problem and the modern mathematical theory of Optimal Transport (OT)[22]. Although both fields aim to solve allocation and distribution problems, they have historically evolved along separate

trajectories. Classical transportation algorithms such as Northwest Corner (NWC), Modified Distribution Method (MODI), and Stepping Stone were developed pragmatically within OR to handle resource allocation efficiently, but they lack the rigorous geometric and duality-based interpretation found in OT. Conversely, OT theory, rooted in the work of Monge and Kantorovich, evolved within functional analysis and geometry, offering a deeper theoretical understanding but often lacking the practical, table-based heuristics widely used in industry. The proposed unified framework synthesizes these two traditions, revealing that they are fundamentally aligned through the language of linear programming, duality, and polyhedral geometry.

At its core, the transportation problem in OR is a special case of Kantorovich's relaxed OT formulation, where discrete mass distributions correspond directly to supply and demand nodes in a transportation tableau[23]. By mapping each OR algorithmic step to geometric movements within the transportation polytope, the unified framework demonstrates that feasible solutions generated by NWC, MODI, and Stepping Stone are not arbitrary heuristics; rather, they correspond to traversals along edges of the OT transport polytope. This reveals that classical methods implicitly exploit the same geometric structures such as extreme points, adjacency relations, and basis cycles that OT theory formalizes rigorously. MODI's reduced cost calculations, for example, align directly with the computation of dual potentials in the Kantorovich dual problem, while Stepping Stone's cycle adjustments correspond to perturbations along minimal faces of the feasible polytope.

Furthermore, the framework exposes the algebraic unity between OR's basis representations and OT's dual formulations. The dual variables in MODI, traditionally used to assess optimality, can be reinterpreted as Kantorovich potentials, which in OT encode the underlying geometry of the cost function. The Stepping Stone method's identification of improvement loops is structurally equivalent to exploring optimal transport plans' support graphs, revealing a deep graph-theoretic connection. In this way, classical OR methods are shown to be computational instantiations of principles that in OT theory are derived from convex analysis, duality, and geometric measure theory.

By framing NWC, MODI, and Stepping Stone within a single algebraic geometric model, the research highlights that these classical algorithms approximate particular solution paths within the OT landscape. This connection not only enriches the theoretical foundation of OR methods but also opens pathways for cross-fertilization[24]. OT concepts such as entropic regularization, Wasserstein metrics, and geometric projections can now be applied to enhance or reinterpret classical transportation algorithms. Meanwhile, practical insights from OR, such as degeneracy handling and computational heuristics, can inform more efficient implementations of OT solvers.

Ultimately, the unified framework dissolves the historical separation between OR and OT, positioning the classical transportation problem as both a precursor and a discrete instantiation of modern transport theory. This synthesis demonstrates that OR algorithms possess latent mathematical structures that align naturally with the geometric and dual formulations of OT. By making these connections explicit, the framework provides a foundation for future methodological innovations that integrate the robustness of mathematical transport theory with the computational pragmatism of classical operations research.

Contributions

This research offers several significant contributions to the study of transportation problems, particularly by establishing a novel bridge between classical Operational Research (OR) techniques and the modern mathematical structure of Optimal Transport (OT). These contributions span theoretical development, methodological innovation, practical implications, and computational insights, thereby positioning the proposed work as a foundational step in reinterpreting traditional algorithms through a rigorous algebraic-geometric lens.

From a theoretical perspective, the study introduces a formal unification of the North-West Corner (NWC) method, MODI (Modified Distribution Method), and the Stepping Stone algorithm within the framework of optimal transport theory. While these classical methods have historically been taught and applied as heuristic or algorithmic procedures, this research demonstrates that they can be reformulated as structured navigations across the vertices, edges, and cycles of the optimal transport

polytope. By embedding these procedures within the duality theory of Kantorovich transport, the research shows that NWC corresponds to a particular primal-feasible vertex selection strategy, MODI operates as a dual-feasible reduced-cost evaluation mechanism, and Stepping Stone implements geometric edge traversal along admissible spanning trees. This unified interpretation establishes, for the first time, a coherent mathematical foundation linking the combinatorial actions of classical OR to the convex-analytic structure of OT.

The methodological contribution lies in the development of a new algebraic-geometric framework that systematically describes transportation algorithms in terms of polytopes, bipartite graphs, basis cycles, and dual potentials[25]. This framework provides a consistent notation, a formal set of lemmas, and a geometric language that jointly characterize primal feasibility, dual feasibility, degeneracy, pivot operations, and optimality conditions. It also recasts algorithmic steps as transformations within a structured search space, allowing classical methods once perceived as procedural heuristics to be interpreted as instances of a broader and mathematically principled optimization architecture. Such a methodological shift enables the construction of new hybrid algorithms, analysis of degeneracy through polytope geometry, and a generalization of optimality conditions using dual potentials analogous to Kantorovich potentials in OT.

The research also yields important practical contributions, particularly for pedagogy, algorithm design, and problem-solving in logistics and supply chain management. By presenting NWC, MODI, and Stepping Stone within a single unified mathematical system, the framework simplifies conceptual understanding and allows students and practitioners to see the inherent relationships between methods that were previously taught as separate tools. This unification clarifies how initial feasible solutions evolve toward optimality, how adjustments propagate through the transportation table, and how cost-efficiency is evaluated in geometric terms[26]. Moreover, the framework highlights potential algorithmic enhancements, such as improved handling of degeneracy, alternative vertex-initialization strategies inspired by OT heuristics, and more robust reduced-cost computations grounded in dual theory. These insights can directly support both educational delivery and practical optimization tasks in industrial applications.

Finally, the study contributes computational insights by analyzing the performance characteristics of classical algorithms when reinterpreted through the structure of OT polytopes. The unified framework allows comparison of traversal paths, identification of redundant operations, and examination of convergence behavior from a geometric standpoint[27]. For instance, the research shows that NWC systematically selects a particular class of vertices that may be suboptimal in terms of traversal length; MODI's reduced-cost evaluations can be mapped to dual-gradient directions within the OT dual space; and Stepping Stone's cycle adjustments correspond to specific adjacency movements along the polytope's one-dimensional faces. These computational interpretations provide a stronger basis for analyzing algorithmic complexity, identifying bottlenecks, and proposing more efficient variants.

4. Conclusion

This research has demonstrated that the classical transportation algorithms Northwest Corner (NWC), MODI, and Stepping Stone can be rigorously unified within a single mathematical framework grounded in Optimal Transport (OT) theory. By reformulating each method through the lens of Kantorovich's relaxation, duality theory, and the geometry of the transportation polytope, the study establishes that these heuristics are not isolated algorithmic procedures but rather specific navigational strategies on the same feasible geometric structure. The results confirm that NWC corresponds to a deterministic initialization on the vertices of the transport polytope, MODI mirrors the adjustment of dual potentials in OT, and the Stepping Stone method represents movement along basis cycles that are homomorphic to adjacency relations in the polytope. Through these equivalences, the study shows that the three classical algorithms reflect different geometric interpretations of primal feasibility, reduced costs, and optimality conditions. Thus, the unified framework not only harmonizes

these algorithms algebraically but also provides a coherent geometric foundation that clarifies their operational logic and mutual relationships. This work contributes significantly to the theoretical discourse by bridging classical OR algorithms with modern OT theory, demonstrating that traditional heuristics can be regarded as low-dimensional approximations of the broader optimal transport machinery. Methodologically, the framework enables more transparent analysis of degeneracy, basis transitions, and optimality verification by referencing well-established structures in OT such as dual potentials, the cost matrix, and the geometric adjacency of feasible solutions. Practically, the unification simplifies how transportation algorithms are taught, understood, and improved, providing a more intuitive explanation for their behavior while also opening new opportunities for algorithmic refinement. Even though the framework introduces deeper mathematical clarity, its complexity may present limitations when generalized to large-scale or continuous optimal transport problems. Nonetheless, this research lays a robust foundation for integrating classical OR methods into the more expansive OT landscape and sets the stage for future work exploring computational enhancements and hybrid algorithms inspired by modern optimal transport theory. Overall, the unified mathematical framework presented in this study not only strengthens the conceptual bridge between historical transportation methods and contemporary transport theory but also offers profound insights that can advance computational practice, pedagogical approaches, and theoretical development in both Operations Research and Optimal Transport.

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